ON THE STABILITY OF MOTION OF CERTAIN TYPES OF GYROSTATS

(OB USTOICHIVOSTI DVIZHENIA GIROSTATOV NEKOTOROGO VIDA)

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In [1] the author investigated certain types of motion of a heavy gyrostat with one point fixed*. This work presents an investigation of the stability of motion of heavy gyrostats of a special kind resting on a fixed horizontal plane.

1. Let the axis of rotation of a symmetric rigid body S_2 (rotor) be fixed in another rigid body S_1 representing a gyrostat. Let us assume further that this axis of rotation coincides with the rotation axis of the central ellipsoid of inertia of the body S_2. The equation of motion of S_2 with respect to S_1 in the Lagrange form is

$$\frac{d}{dt}\frac{\partial T_2}{\partial \alpha'} - \frac{\partial T_2}{\partial \alpha} = Q \tag{1.1}$$

Here T_2 is the kinetic energy of S_2 in its motion relative to the fixed coordinate system $\xi\eta\zeta$; α is the angle between two planes through the axis of rotation, of which one is fixed in S_1 and another in S_2; Q is the generalized force corresponding to the coordinate α.

By Koenig's theorem we have

$$2T_2 = m_2 v_2^2 + A_2(\omega_1^2 + \omega_2^2) + C_2(\omega_3 + \alpha')^2$$

where m_2, $A_2 = B_2$, C_2 are, respectively, the mass and the principal central moments of inertia of the body S_2; v_2 is the velocity vector of

* It should be mentioned here that in [1] on p.11, last line, K_1 = const should read k_1 = const. [Page and line numbering refers to translated *PMM*.]

its center of gravity O_2; ω_i (i = 1, 2, 3) are the components along the principal central axes of inertia of S_2 of the velocity vector of S_1 moving relative to the fixed coordinate system. Equation (1.1) now becomes

$$C_2(\omega_3' + \alpha'') = Q \tag{1.2}$$

We shall assume that the only force which acts on the rotor is the pressure of S_1 at the end parts of its axis. In this case Q = 0 and Equations (1.2) yield immediately the first integral

$$\omega = \omega_3 + \alpha' = \text{const} \tag{1.3}$$

expressing the fact that the component of the angular velocity along the rotor's axis of rotation is constant [3].

The angular momentum **G** of the gyrostat S with respect to an arbitrary point is the vector sum of the angular momenta of the rigid body S_1 and S_2. The angular momentum of S_2 about its center of gravity O_2 equals

$$A_2\mathbf{e} + C_2\omega\mathbf{k} \qquad (\mathbf{e}^2 = \omega_1^2 + \omega_2^2)$$

Here **k** is the unit vector along the rotor's axis; **e** is the equatorial component of the rotor's angular velocity.

Following Zhukovskii [2] we introduce an auxiliary rigid body consisting of the body S_1 and of an infinitely thin rod of mass m_2 connected rigidly with S_1 whose center of gravity is at O_2. The rod is directed along the rotor's axis, and its moment of inertia about a vertical axis through O_2 is A_2. The angular momentum vector **G** equals the sum of the vector $C_2\omega\mathbf{k}$ plus the angular momentum vector of the auxiliary body. Let us introduce the coordinate system $Oxyz$ whose axes are along the principal axes of inertia of the auxiliary body and the origin O is in S_1. The x-, y- and z-components of the angular momentum vector **G** are

$$G_x = Ap + k_1, \qquad G_y = Bq + k_2, \qquad G_z = Cr + k_3 \tag{1.4}$$

Here A, B, C are the moments of inertia of the auxiliary body; p, q, r are the components of the angular velocity vector of the body S_1 along the moving axes; k_i (i = 1, 2, 3) are the components of the vector $C_2\omega\mathbf{k}$ also along the moving axes. We can conclude now that if the gyrostat has a fixed point O, then its equations of motion caused by the gravity forces are in the form (1.1) of [1], and the results of the investigations of stability of gyrostats when k_i = const given in [1] are applicable also to our body S_1 with its rotor S_2.

2. We shall examine now the motion of the Gervat gyroscope (named by Gervat "Pied équilibriste") [3], resting on an absolutely smooth horizontal plane, and acted upon only by gravity forces.

The device consists of the body S_1 in the form of a semicircular frame *ABC*, whose circular segment is directed downward, connected rigidly to the leg *BDE*, and of the symmetric rotor S_2. The frame *ABC* supports the rotor's axis in the bearings at *A* and *C*. The leg *BDE* has a rectilinear part *DE* which is perpendicular to the plane of the semicircular frame *ABC*, enabling the device to stand on the horizontal plane. The middle plane of the rotor passes through *DE*, the center of gravity of the rotor O_2 is on the axis *AC* in the plane through *DE* perpendicular to *AC*.

When the rotor is not moving, then the vertical position of equilibrium of the device is unstable; when it rotates sufficiently fast then the equilibrium is stable. The theory of the Gervat gyroscope has been given by Carvallo, who assumed that the body S_1 is weightless; otherwise the theory becomes considerably more complicated.

Let $\xi\eta\zeta$ be the fixed coordinate system, the axes ξ and η being in the horizontal plane, and the ζ-axis being directed vertically upwards. Let the coordinate system $Oxyz$ move with the body S, its origin O coinciding with the center of gravity of the whole device, which is located on the line BO_2. The line OO_2 is directed upwards and coincides with the y-axis; the x-axis, perpendicular to the y-axis, is in the middle plane of the rotor, and the z-axis perpendicular to the x- and y-axes forms with them a right-handed system. The z-axis is obviously parallel to the rotor's axis *AC*.

Let the x-, y- and z-axes be the principal central axes of inertia of the auxiliary body. Let us consider also the coordinate system $Ox_1y_1z_1$ whose axes are parallel, respectively, to the axes ξ, η, ζ of the fixed system. Let the coordinates of the center of mass O in the fixed system be ξ, η, ζ, and let the angle between the z_1- and z-axes be θ, and the angle between the x_1- and x-axes be ψ. The length of the normal OP which is dropped from the point O to the base of *DE* is l.

Clearly

$$\zeta = l \sin\theta \qquad (2.1)$$

The x-, y- and z-components of the instantaneous angular velocity of the body S_1 equal, respectively

$$p = \theta', \quad q = \psi' \sin\theta, \quad r = \psi' \cos\theta \qquad (2.2)$$

Since the forces acting on the device, that is the weight Mg and the normal reaction of the plane N, are vertical, the point Q which is the projection of the point O on the horizontal plane, in general, will move uniformly in a straight line. This would occur in a general case, but we can assume without any loss of generality that the point Q remains stationary.

Since in our case $k_1 = k_2 = 0$, $k_3 = C_2\omega$, we can use general theorems of mechanics to derive the following energy integrals:

$$M\left(\frac{d\zeta}{dt}\right)^2 + Ap^2 + Bq^2 + Cr^2 + 2Mg\zeta = h \qquad (2.3)$$

and area integrals

$$Bq \sin\theta + (Cr + C_2\omega)\cos\theta = k \qquad (2.4)$$

Here M is the mass of the device, g is the gravitational acceleration, h and k are arbitrary constants.

Substituting into (2.3) and (2.4) for p, q, r, ζ their equivalents given in Formulas (2.1) and (2.2), we obtain the equation

$$(b + e\cos^2\theta)(1 + c\cos^2\theta)\,\theta'^2 = (\alpha - a\sin\theta)(1 + c\cos^2\theta) - (\beta - c_2\omega\cos\theta)^2 \qquad (2.5)$$

Here

$$a = \frac{2Mgl}{B}, \quad b = \frac{A}{B}, \quad c = \frac{C-B}{B}, \quad c_2 = \frac{C_2}{B}, \quad e = \frac{Ml^2}{B}, \quad \alpha = \frac{h}{B}, \quad \beta = \frac{k}{B}$$

The integration of Equation (2.5) results in a formula for t in the form of a hypergeometric integral of θ. Inversion of this integral reduces the calculations of the angles ψ and α to quadratures on account of

$$\psi' = \frac{\beta - c_2\omega\cos\theta}{1 + c\cos^2\theta}, \qquad \alpha' = \omega - \psi'\cos\theta$$

We could also find the curve traced on the horizontal plane by the point P of the base DE [3].

3. We shall consider now a gyrostat S which will differ from the device investigated in Section 2 by the design of its leg. Let the leg of the gyrostat frame have a plane knife-like stand in the form of a circular segment [4] of radius a. The segment's center O_1 is on the y-axis, and the y-coordinate of O_1 equals a_1. This design adds one more degree of freedom; the device can rock about a horizontal axis perpendicular to the chord of the circular segment.

The orientation of the moving coordinate system $Oxyz$ relative to the fixed coordinate system $Ox_1y_1z_1$ will be determined through the Eulerian angles ϑ, ψ, ϕ. Instead of Formulas (2.1) and (2.2) we have now

$$\zeta = a\sin\theta - a_1\sin\theta\cos\varphi \qquad (3.1)$$

$$p = \psi'\sin\theta\sin\varphi + \theta'\cos\varphi, \quad q = \psi'\sin\theta\cos\varphi - \theta'\sin\varphi, \quad r = \varphi' + \psi'\cos\theta \qquad (3.2)$$

The energy integral retains its form (2.3), but the integral of the areas becomes

$$Ap \sin\theta \sin\varphi + Bq \sin\theta \cos\varphi + (Cr + C_2\omega)\cos\theta = \text{const} \tag{3.3}$$

We shall investigate the stability of equilibrium of the gyrostat's vertical position occurring at the following values of the variables:

$$\theta = {}^1/_2\pi, \quad \theta' = 0, \quad \psi = \text{const}, \quad \psi' = 0, \quad \varphi = \varphi' = 0, \qquad \omega = \text{const} \tag{3.4}$$

In the perturbed motion we set

$$\theta = {}^1/_2\pi + \theta_1$$

retaining the same notation for the remaining variables.

By Equations (3.1) and (3.2), the integrals (2.3) and (3.3) will take on for the perturbed motion the following form:

$$\begin{aligned} V_1 &= A\theta_1'^2 + B\psi'^2 + C\varphi'^2 + Mg\,(a_1\varphi^2 - l\theta_1^2) + \ldots = \text{const} \\ V_2 &= (A - B)\,\theta_1'\varphi + B\psi' - (C\varphi' + C_2\omega)\,\theta_1 + \ldots = \text{const} \end{aligned} \tag{3.5}$$

Here $l = a - a_1$, and the dots indicate the omitted terms of the third and higher order of smallness. Let us consider the function

$$\begin{aligned} V = V_1 + \lambda V_2^2 = A\theta_1'^2 + C\varphi'^2 + Mga_1\varphi^2 + \\ + B\,(1 + \lambda B)\,\psi'^2 - 2\lambda BC_2\omega\,\theta_1\psi' + (C_2^2\omega^2\lambda - Mgl)\,\theta_1^2 + \ldots \end{aligned} \tag{3.6}$$

where λ is a certain constant.

By Sylvester's criterion, the necessary and sufficient condition for the function V to be positive-definite with respect to the variables under consideration is

$$1)\quad 1 + \lambda B > 0, \qquad 2)\quad \lambda\,(C_2^2\omega^2 - BMgl) - Mgl > 0, \qquad 3)\quad a_1 > 0$$

Clearly, the above inequalities can always be satisfied by an appropriate choice of the quantity $\lambda > 0$ if

$$a_1 > 0, \qquad C_2^2\omega^2 - BMgl > 0 \tag{3.7}$$

When the conditions (3.7) are satisfied, then the function V is positive-definite, and its time derivative, since we are dealing with the perturbed motion, equals zero. Thus the function V satisfies all conditions of Liapunov's theorem on stability.

In this way, the conditions (3.7) turn out to be the sufficient conditions of stability of the gyrostat's unperturbed motion (3.4) with respect to the variables θ, θ', ψ', ϕ, ϕ'. The existence of the integral

(1.3) makes the motion (3.4) stable with respect to the variable α' as well.

It can be easily shown that the inequalities (3.7) are the necessary conditions of stability for the gyrostat's vertical position of equilibrium as well. Let us consider the equations with variations for the perturbed motion

$$A\theta_1'' + C_2\omega\psi' = Mgl\theta_1, \qquad B\psi'' - C_2\omega\theta' = 0$$

The characteristic equation of these equations

$$\Delta(\lambda) = \lambda^2 (AB\lambda^2 + C_2{}^2\omega^2 - BMgl) = 0$$

in the case when $C_2{}^2\omega^2 - BMgl < 0$, has a root with a positive real part, which indicates that in this case the perturbed motion is unstable.

The condition $a_1 > 0$ indicates that the geometric center of the stand, which has the form of a circular segment, should be located above the center of gravity of the device. This condition is the necessary and sufficient condition of stability of rocking the gyrostat about an axis in the plane of the base perpendicular to the chord of the segment [4].

The second condition in (3.7) allows the determination of the smallest angular velocity of the rotor S_2 at which the gyrostat is still stable.

It is interesting to compare the above condition with Maievskii's condition of the rotational stability of a gyroscope

$$C^2\omega^2 - 4AMgl > 0$$

We notice that, other conditions being equal, the stability of a gyrostat is established at an angular velocity which is half as large as that of a gyroscope.

4. We shall consider now the stability of motion of a gyrostat S_1 with a spherical base resting on a horizontal plane. The gyrostat S_1 housing the rotor S_2 can, for example, be in the shape of a hollow sphere, like the gyroscopic sphere of Bobylev [5]. The spherical base of S_1 touches the supporting plane at the point P. Let the geometric center of the spherical base O_1 not coincide with the center of gravity of the gyrostat O, let the moving coordinate system $Oxyz$ have its origin at O and let its axes be along the principal axes of the central ellipsoid of inertia of the auxiliary body, which itself is an ellipsoid of revolution about the Oz-axis. Let the point O_1 be on the Oz-axis, let its z-coordinate be a_1, let the radius of the spherical base be a, and let the axis of the rotor S_2 coincide with the Oz-axis.

The equations of motion of a heavy gyrostat resting on a fixed horizontal plane are derived from the general theorems of mechanics. From the theorem on the motion of the center of gravity of a system we have the following equation:

$$\begin{aligned} \frac{du}{dt} + qw - rv &= X - g\gamma_1 \\ \frac{dv}{dt} + ru - pw &= Y - g\gamma_2 \\ \frac{dw}{dt} + pv - qu &= Z - g\gamma_3 \end{aligned} \tag{4.1}$$

Here u, v, w are, respectively, the x-, y-, z-components of the velocity vector of the center of gravity O; X, Y, Z are the components of the reaction of the fixed plane at the point of contact P caused by a unit mass. The direction cosines of the ζ-axis with respect to the axes x, y, z are γ_1, γ_2, γ_3, respectively, and they satisfy the Poisson equation

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \qquad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \qquad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2 \tag{4.2}$$

On the strength of the theorem of the angular momentum of a system about its center of gravity we have

$$\begin{aligned} A\frac{dp}{dt} + (C - A)\, qr + C_2\omega\, q &= M\,(yZ - zY) \\ A\frac{dq}{dt} + (A - C)\, pr - C_2\omega p &= M\,(zX - xZ) \\ C\frac{dr}{dt} &= M\,(xY - yX) \end{aligned} \tag{4.3}$$

where the coördinates of the point of contact P are

$$x = -a\gamma_1, \qquad y = -a\gamma_2, \qquad z = a_1 - a\gamma_3 \tag{4.4}$$

The equations of motion of a gyrostat (4.1), (4.2), (4.3) must be accompanied by constraint equations. If the supporting horizontal plane is absolutely rough, then the velocity of the point of contact P equals zero, that is

$$u + qz - ry = 0, \qquad v + rx - pz = 0, \qquad w + py - qx = 0 \tag{4.5}$$

If the plane is not absolutely rough, then the body S_1 can slide. The friction force F is proportional to the normal reaction N, and it opposes the motion of the point of contact P. The elementary work of the reaction in a real displacement is obviously non-positive:

$$X\,(u + qz - ry) + Y\,(v + rx - pz) + Z\,(w + py - qx) \leqslant 0 \tag{4.6}$$

If the horizontal plane is absolutely smooth, then the reaction is normal to the plane, and also

$$X = N\gamma_1, \qquad Y = N\gamma_2, \qquad Z = N\gamma_3 \tag{4.7}$$

In every case the velocity vector of the point of contact P is perpendicular to the ζ-axis, hence the following equation

$$\gamma_1 (u + qz - ry) + \gamma_2 (v + rx - pz) + \gamma_3 (w + py - qx) = 0 \tag{4.8}$$

must be satisfied.

Let us multiply Equations (4.1) by Mu, Mv, Mw, Equations (4.3) by p, q, r, respectively, and add them; the result is

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}[M(u^2 + v^2 + w^2) + A(p^2 + q^2) + Cr^2 + 2Mg\zeta] \\ = M[X(u + qz - ry) + Y(v + rx - pz) + Z(w + py - qx)]\end{gathered}$$

From the above equations and from the relations (4.5) to (4.8) follows

$$M(u^2 + v^2 + w^2) + A(p^2 + q^2) + Cr^2 + 2Mg\zeta \leqslant \text{const} \tag{4.9}$$

In the case of an absolutely smooth or an absolutely rough surface we shall have only the signs of equality indicating the existence of the energy integral.

It is easily seen that the height of the center of gravity of the gyrostat above the plane is $\zeta = a - a_1\gamma_3$.

Let us now multiply Equations (4.3) by the coordinates of the point of contact P, that is by x, y, z, respectively, and add them. By Equation (4.4) and (4.2) we obtain the first integral

$$A(px + qy) + (Cr + C_2\omega) z = \text{const} \tag{4.10}$$

The integral (4.10) shows that the scalar product of the gyrostat's angular momentum vector about its center of gravity O multiplied by the radius vector of the point P centered on O, is constant. In the case when $a_1 = 0$, the gyrostat's center of gravity O coincides with the center of the spherical base O_1, and the integral (4.10) becomes the area integral.

Poisson's equations (4.2) possess the obvious integral

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{4.11}$$

In the case of an absolutely smooth surface, from the third equation in (4.3) and by (4.4) and (4.7) follows the first integral

$$r = \text{const} \tag{4.12}$$

We shall investigate now the stability of the gyrostat on a rough or on a smooth surface. The equations of motion (4.1) to (4.3) have the following particular solution:

$$p = q = 0, \quad r = r_0, \quad u = v = w = 0, \quad \gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1 \tag{4.13}$$

whereby

$$X = Y = 0, \quad Z = g, \quad x = y = 0, \quad z = a_1 - a = -l$$

In the perturbed motion we set $r = r_0 + \beta_1$, $\gamma_3 = 1 + \beta_2$ and leave the remaining variables as they were.

By the equations of the perturbed motion we have

$$\begin{gathered} V_1 = M(u^2 + v^2 + w^2) + A(p^2 + q^2) + C(\beta_1^2 + 2r_0\beta_1) - 2Mga_1\beta_2 \leqslant V_{10} \\ V_2 = A(p\gamma_1 + q\gamma_2) + (Cr_0 + C_2\omega)\beta_2 + C\beta_1\beta_2 + C\frac{l}{a}\beta_1 = \text{const} \\ V_3 = \gamma_1^2 + \gamma_2^2 + \beta_2^2 + 2\beta_2 = 0 \end{gathered} \tag{4.14}$$

Here V_{10} = const represents the initial value of the function V_1; the equality sign occurs when the surface is ideally rough or smooth, and the inequality signs occur when there is sliding with frictional forces opposing the motion of the point of contact.

Let us consider the function

$$\begin{aligned} V &= V_1 + 2\lambda V_2 + \mu V_3 + \tfrac{1}{4}(C - A)\lambda^2 V_3^2 \\ &= A(p^2 + q^2) + 2A\lambda(p\gamma_1 + q\gamma_2) + \mu(\gamma_1^2 + \gamma_2^2) + \\ &+ C\beta_1^2 + 2C\lambda\beta_1\beta_2 + [(C - A)\lambda^2 + \mu]\beta_2^2 + \\ &+ M(u^2 + v^2 + w^2) + 2C\left(r_0 + \lambda\frac{l}{a}\right)\beta_1 + \frac{1}{2}(C - A)\lambda^2(\gamma_1^2 + \gamma_2^2 + \beta_2^2)\beta_2 \end{aligned} \tag{4.15}$$

where λ is a constant, and $\mu = Mga_1 - (Cr_0 + C_2\omega)\lambda$. By Sylvester's criterion the necessary and sufficient condition for the quadratic form appearing in the function V to be positive-definite is the inequality

$$f(\lambda) \equiv A\lambda^2 + (Cr_0 + C_2\omega)\lambda - Mga_1 < 0 \tag{4.16}$$

In general, the above inequality is satisfied when the polynomial $f(\lambda)$ has two distinct real roots, that is when

$$(Cr_0 + C_2\omega)^2 + 4AMga_1 > 0 \tag{4.17}$$

It is quite clear that the linear part of the function V vanishes when

$$\lambda = -\frac{a}{l} r_0$$

and with the above value of λ the condition (4.16) assumes the form

$$\left(C - A\ \frac{a}{l}\right) r_0^2 + C_2\omega\, r_0 + Mg\,\frac{a_1 l}{a} > 0 \qquad (4.18)$$

Since $dV/dt = dV_1/dt \leqslant 0$, the function (4.15) subjected to the condition (4.18) satisfies all the conditions of Liapunov's theorem on stability. In this way, the inequality (4.18) becomes the gyrostat's sufficient condition of stability of the unperturbed motion (4.13), with respect to the quantities u, v, w, p, q, r, γ_1, γ_2, γ_3.

If we set $\omega = 0$, then the inequality (4.18) assumes the form

$$\left(C - A\,\frac{a}{l}\right) r_0^2 + Mg\,\frac{a_1 l}{a} > 0 \qquad (4.19)$$

and becomes the sufficient condition of stability of rotation about a vertical axis of a heavy solid with a spherical base resting on a horizontal plane.

When $a_1 < 0$, then the center of gravity of the body is above the geometric center of the spherical base. In this case the rotation of the body is stable if the rotational angular velocity is sufficiently great and if the condition $Cl > Aa$ is satisfied.

On the other hand, when $a_1 > 0$, then the center of gravity of the body is below the geometric center O_1. In this case, when $Cl > Aa$ is satisfied, the rotation of the body is stable for all values of the angular velocity r_0; if the last inequality is not satisfied, that is, when $Cl < Aa$, then the rotation may still be stable when the additional condition

$$r_0^2 < \frac{Mg}{Aa - Cl}\,\frac{a_1 l^2}{a}$$

is satisfied.

The above analysis of stability is applicable in particular to the tippe-top ($a_1 > 0$), which has been investigated in numerous works. In particular, in [6] we have the investigation of the motion of such a top in the first approximation and on the assumption that at the point of contact of the top with the surface forces of viscous friction are acting. The authors of the above-mentioned paper, using linearized equations of motion, have found the necessary and sufficient conditions of stability of motion of the top about a vertical axis, which agrees with our condition (4.19), except for notation.

If the surface is absolutely smooth, then the equations of the perturbed motion have not only the first integrals (4.14) but also the first integral

$$V_4 = \beta_1 = \text{const}$$

We construct the function

$$W = V - 2C\left(r_0 + \lambda \frac{l}{a}\right) V_4$$

where V is determined in (4.15), and find that in this case the sufficient condition of stability of the unperturbed motion (4.13) is the inequality (4.17).

We shall investigate also the function

$$W_1 = A\,(p\gamma_2 - q\gamma_1)$$

whose time derivative, on the strength of the equations of the perturbed motion, equals

$$W_1' = A\,(p^2 + q^2) - (Cr_0 + C_2\omega)\,(p\gamma_1 + q\gamma_2) - Mga_1\,(\gamma_1^2 + \gamma_2^2) + \ldots$$

where the dots indicate the omitted terms of higher order of smallness. It is clear that under the condition

$$(Cr_0 + C_2\omega)^2 + 4AMga_1 < 0$$

the function W_1' is positive-definite. By Chetaev's theorem on instability the unperturbed motion (4.13) is unstable.

We conclude that the inequality (4.17) is the necessary and sufficient condition of a gyrostat's stability of rotation about a vertical axis on an absolutely smooth horizontal plane.

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